Renormalization group scale-setting from the action - a road to modified gravity theories

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Abstract

The renormalization group (RG) corrected gravitational action in Einstein-Hilbert and other truncations is considered. The running scale of the renormalization group is treated as a scalar field at the level of the action and determined in a scale-setting procedure recently introduced by Koch and Ramirez for the Einstein-Hilbert truncation. The scale-setting procedure is elaborated for other truncations of the gravitational action and applied to several phenomenologically interesting cases. It is shown how the logarithmic dependence of the Newton's coupling on the RG scale leads to exponentially suppressed effective cosmological constant and how the scale-setting in particular RG corrected gravitational theories yields the effective $f(R)$ modified gravity theories with negative powers of the Ricci scalar R. The scale-setting at the level of the action at the non-gaussian fixed point in Einstein-Hilbert and more general truncations is shown to lead to universal effective action quadratic in Ricci tensor.

1 Introduction

The discovery of the fascinating phenomenon of late-time accelerated cosmic expansion and detailed studies of effects attributed to dark matter have given a new impetus to investigation of quantum effects on curved background as a possible source of their explanation. Quantum Field Theory (QFT) in curved space-time studies the connection of quantum phenomena and the background metric, allowing the investigation of influence of quantum fluctuations on the dynamics of space-time. Running (scale-dependent) parameters of the theory are one of principal

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results in QFT in curved space-time. At least two directions of research centered on the role of running parameters in the study of gravitational phenomena in cosmology and astrophysics have been developed in the last decade or so. The first is related to perturbative radiative corrections as a source of the scale dependence of the parameters of the theory [1, 2, 3, 4, 5, 6]. The second one is based on the concept of the average effective action accompanied by the machinery of the exact renormalization group [7, 8, 9]. Some of the mentioned references also tried to develop methods useful for both approaches with scale dependent parameters [4, 6]. In this paper we follow a similar philosophy.

A serious issue behind many, if not all, modified gravity theories is the fundamental origin of corrections to General Relativity (GR). Recently there have been several proposals how to relate the renormalization group (RG) effects to corrections to GR and modification of cosmological dynamics in general [10, 11, 12, 13, 14, 15]. Our main goal in this paper is to systematically establish a connection between the QFT in curved space-time with its running (scale-dependent) parameters and modified gravity theories. We base our approach on a procedure of scale-setting at the level of the action recently introduced by Koch and Ramirez [16]. Our contributions beyond [16] presented in this paper comprise a technical extension of the scale-setting procedure to the action with higher-derivative terms (section 2.2), a possible explanation of the smallness of the observed cosmological constant (section 3), a demonstration how the scale-setting procedure may result in $1/R^{\alpha}$ terms in the effective action (section 4) and the description of the universal form of the effective action at the non-gaussian fixed point in Einstein-Hilbert and other truncations (section 5).

2 Scale-setting in quantum-corrected gravitational action

The gravitational part of the action in general relativity with cosmological constant is described by the well-known Einstein-Hilbert action:

$$S_{EH} = \int d^4x \sqrt{-g} \frac{1}{16\pi G} (R - 2\Lambda) \,, \tag{1}$$

with $\Lambda = 8\pi G \rho_\Lambda$. This action should be supplemented with the matter action S_m, i.e. the total action is $S = S_{EH} + S_m$. Quantum corrections to this action, emanating from the fluctuations of quantum fields, modify the Einstein-Hilbert action in two principal ways: 1) additional higher-order terms in curvature must be added to remove the divergences in the process of renormalization and 2) the parameters of the terms in this enlarged action (such as G and Λ) acquire scale dependence. The scale-dependent parameters can be introduced at the level of solutions, if they are available in analytical form, at the level of the EOM [4, 5, 6, 17] or at the level of the action. In this paper we concentrate on the last of these approaches.

Once the level where the scale-dependent (or running) parameters will be introduced is selected, an essential question is which value should be taken for the scale itself. As various arguments have been put forward in favor of different choices

for the scale, a clear need for a systematic scale-setting procedure exists. For the approach based on correcting the equations of motion, the scale-setting procedure was proposed and applied in cosmology several years ago [4], with a recent extension to space-times of lower symmetry with applications for astrophysical systems [6]. The scale-setting procedure at the level of the action, recently proposed in [16], opens ways for additional insights into the workings of quantum corrected gravitational theories.

2.1 Einstein-Hilbert truncation

In the Einstein-Hilbert truncation of the vacuum action of gravity, the parameters G and Λ become running parameters G_k and Λ_k. i.e. they acquire scale dependence. In the approach of [16], the running parameters are introduced at the level of the action:

$$S_{EH} = \int d^4x\sqrt{-g}\frac{1}{16\pi G_k}(R - 2\Lambda_k)\,. \tag{2}$$

The identification of the scale obtained in the scale-setting procedure should be consistent with the original role of the scale, e.g. an infrared cutoff of an average effective action for the theory of gravity. The proof of this consistency might not be easily feasible. However, any other choice for scale other than the one obtained in the scale-setting procedure might be at odds with general covariance or require phenomenological extensions such as energy-momentum interchange with matter components. The scale-setting procedure represents a self-consistent way of determining the scale k.

The most general choice for the scale k respecting the requirement of general covariance is for k to become a scalar filed i.e. $k \to k(x)$, as proposed in [16]. This assumption has been recently incorporated into a number of lines of research [14, 15] in which an Ansatz for the identification of k with some curvature invariants was used. A crucial advantage of the scale-setting procedure described below is that, once the functional dependences of the action parameters on k are known, the identification of the scale follows without additional assumptions.

Following the presentation of [16], the variation of the action (2) with respect to $k(x)$ yields the relation

$$R\left(\frac{1}{G_k}\right)' - 2\left(\frac{\Lambda_k}{G_k}\right)' = 0\,, \tag{3}$$

where primes denote differentiation with respect to k. The variation of the action (2) with respect to metric $g^{\mu\nu}$ yields an equation of motion

$$G_{\mu\nu} = -8\pi G_k T_{\mu\nu} - \Lambda_k g_{\mu\nu} - \Delta t_{\mu\nu}\,, \tag{4}$$

where $G_{\mu\nu} = R_{\mu\nu} - 1/2Rg_{\mu\nu}$ is the Einstein tensor and $\Delta t_{\mu\nu} = G_k(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\frac{1}{G_k}$. Taking the covariant derivative of (4) and assuming the conservation of the energy-momentum tensor of matter $\nabla^\nu T_{\mu\nu} = 0$, one obtains the relation

$$R\nabla_\mu\left(\frac{1}{G_k}\right) - 2\nabla_\mu\left(\frac{\Lambda_k}{G_k}\right) = 0\,, \tag{5}$$

which leads directly to (3), using the scalar nature of k.

In this approach, from (3) the curvature scalar R can be expressed as a function of the scale k. The inversion of this function gives the scale k as a function of R. When the expression $k = k(R)$ is inserted into the gravitational field equation (4), the obtained equations of motion resemble those falling to class of $f(R)$ theories[1] (see [18] and references therein). Alternatively, one can insert the result for $k(R)$ into the action (2) to obtain a $f(R)$ modified action. The self-consistent determination of the scale k through a scale-setting procedure represents a way to systematically introduce the modifications of gravity to equations of motion (4) or the action (2).

2.2 The action with higher-order terms

For the reasons of renormalizability, the Einstein-Hilbert action in general needs to be completed with the higher-derivative terms to obtain the vacuum action of QFT in curved space-time [19]. Making a step beyond [16], these additional contributions to the action

$$S_{HD} = \int d^4x\sqrt{-g}(a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2) \tag{6}$$

comprise the terms with the square of the Weyl term $C^2 = R^{\mu\nu\lambda\tau}R_{\mu\nu\lambda\tau} - 2R^{\mu\nu}R_{\mu\nu} + (1/3)R^2$, Gauss-Bonnet term with the integrand $E = R^{\mu\nu\lambda\tau}R_{\mu\nu\lambda\tau} - 4R^{\mu\nu}R_{\mu\nu} + R^2$ and terms with $\Box R$ and R^2. An interesting study of the issue of ghosts in the action (6) was recently given in [20]. The inclusion of quantum corrections arising in QFT makes the coefficients scale-dependent i.e. $a_i \to a_{i,k}$ for $i = 1, 2, 3, 4$. Using the same argumentation as above, we can assume the scale k to be a scalar field and the variation of the total action $S = S_{EH} + S_{HD}$ with respect to the scale k then becomes

$$\frac{1}{16\pi}\left(\left(\frac{1}{G_k}\right)' R - 2\left(\frac{\Lambda_k}{G_k}\right)'\right) + a'_{1,k}C^2 + a'_{2,k}E + a'_{3,k}\Box R + a'_{4,k}R^2 = 0\,, \tag{7}$$

where $'$ denotes differentiation with respect to k. This equation formally connects the scale k with the invariants R, R^2, $\Box R$, C^2 and E. Solving for k in terms of the said invariants and introducing this k-dependence into the gravitational equations of motion obtained by variation of $S = S_{EH} + S_{HD}$ with respect to $g_{\mu\nu}$, one obtains the equations of motion that resemble those of modified gravity theory with the terms in action dependent on R, R^2, $\Box R$, C^2 and E. It is interesting to notice that the variation of E term in (6) does not reduce to total derivative owing to space-time dependence of k, and consequently of $a_{2,k}$.

[1]This is expected since the theory where k is a scalar field is equivalent to Brans-Dicke theory with a potential and $\omega = 0$

3 Scale-setting for a small effective cosmological constant

In the considerations of the running of ρ_Λ and G from QFT in curved space-time [5], the running of the cosmological constant energy density has the quadratic dependence on the scale

$$\rho_{\Lambda,k} = c_0 + c_2 k^2 \,, \tag{8}$$

where c_0 and c_2 are real constants, whereas the Newton's coupling runs logarithmically with the scale

$$G_k = \frac{G_0}{1 + d_2 \ln \frac{k^2}{k_0^2}} \,. \tag{9}$$

Here d_2 and G_0 are real constants. An insertion of (8) and (9) into the scale-setting relation (3) readily yields a simple expression for the scale in terms of the Ricci scalar

$$k^2 = \frac{d_2}{16\pi c_2 G_0} R \,. \tag{10}$$

With this value of the scale, the expressions for the CC energy density and the running Newton's coupling become

$$\rho_{\Lambda,k} = c_0 + c_2 \chi R \,, \quad G_k = \frac{G_0}{1 + d_2 \ln \frac{R}{R_0}} \,, \tag{11}$$

where $\chi = d_2/(16\pi c_2 G_0)$. It is intriguing that the expression for G in (11) has been recently proposed on the basis of analogy with α_{QCD} running coupling constant in [21]. Here it is derived starting from general arguments from QFT on curved space-time using the scale-setting procedure.

An interesting aspect of relations (11), i.e. their dependence on R, is the de Sitter regime that they allow in the maximally symmetric space-time. Namely, in the FLRW space-time, the 00 component of the equation (4) takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_k}{3} \left(\rho_{r,0} \left(\frac{a}{a_0}\right)^{-4} + \rho_{m,0} \left(\frac{a}{a_0}\right)^{-3} \right) + \frac{\Lambda_k}{3} - \frac{\kappa}{a^2} + \frac{\dot{G}_k}{G_k} \frac{\dot{a}}{a} \,. \tag{12}$$

Asymptotically when $a \to \infty$, using $R = 12H^2 + 6\dot{H}$ in the spatially flat FLRW metric, the above equation acquires the form

$$H^2 = \frac{\Lambda_k}{3} + H \frac{\dot{G}_k}{G_k} \,. \tag{13}$$

To have a de Sitter regime, $H^2 = \text{const}$, $R = 12H^2 = \text{const}$, (13) must take the form $H^2 = \frac{8\pi}{3} G_k \rho_{\Lambda,k}$. Inserting the expressions for $\rho_{\Lambda,k}$ and G_k one obtains

$$H^2 = \frac{\frac{8\pi G_0}{3} c_0 + 2 d_2 H^2}{1 + d_2 \ln \frac{12H^2}{R_0}} \,. \tag{14}$$

If c_0 is negligible and $H^2 \neq 0$, (14) gives

$$H^2 = \frac{R_0}{12} e^{\frac{2d_2 - 1}{d_2}} \,. \tag{15}$$

For $d_2 \ll 1$, the de Sitter scale H is exponentially suppressed compared to the scale R_0, as already identified in [21].

4 Power corrections in R

A very interesting question is if the running laws for $\rho_{\Lambda,k}$ or G_k in terms of the scale k can be found which, by identifying k in terms of R through a scale-setting relation, in the Einstein-Hilbert truncation lead to power law corrections in R [22]. In this section we show that such running laws acquire relatively simple forms. For the running laws of the form

$$\rho_{\Lambda,k} = A_1 + B_1 k^{\gamma} \,, \quad \frac{1}{G_k} = A_2 + B_2 k^{\delta} \,, \tag{16}$$

the scale-setting relation (3) yields

$$k = \left(\frac{1}{16\pi} \frac{\delta}{\gamma} \frac{B_2}{B_1} R \right)^{\frac{1}{\gamma-\delta}} \,. \tag{17}$$

Inserting this result for the scale k into (16) gives

$$\rho_{\Lambda,k} = A_1 + B_1 \left(\frac{1}{16\pi} \frac{\delta}{\gamma} \frac{B_2}{B_1} \right)^{\frac{1}{1-\delta/\gamma}} R^{\frac{1}{1-\delta/\gamma}} = A_1 + B_1 \left(\frac{1}{16\pi} \frac{\alpha+1}{\alpha} \frac{B_2}{B_1} \right)^{-\alpha} \frac{1}{R^{\alpha}} \,, \tag{18}$$

$$\frac{1}{G_k} = A_2 + B_2 \left(\frac{1}{16\pi} \frac{\delta}{\gamma} \frac{B_2}{B_1} \right)^{\frac{\delta/\gamma}{1-\delta/\gamma}} R^{\frac{\delta/\gamma}{1-\delta/\gamma}} = A_2 + 16\pi \frac{\alpha}{\alpha+1} B_1 \left(\frac{1}{16\pi} \frac{\alpha+1}{\alpha} \frac{B_2}{B_1} \right)^{-\alpha} \frac{1}{R^{\alpha+1}} \,, \tag{19}$$

where we used $\frac{1}{1-\delta/\gamma} = -\alpha \Rightarrow \frac{\delta}{\gamma} = \frac{\alpha+1}{\alpha}$. For $\alpha > 0$ this is possible if $\delta/\gamma > 1$. There are two ways how this requirement on the ratio δ/γ can be satisfied: $\delta > \gamma > 0$ and $\delta < \gamma < 0$. It is important to notice that the negative powers of R in the effective modified gravity action can be obtained for positive powers of k in (16), i.e. for positive γ and δ. With the identification $A_1 = \rho_{\Lambda}^{*}$, $C = B_1 \left(\frac{1}{16\pi} \frac{\alpha+1}{\alpha} \frac{B_2}{B_1} \right)^{-\alpha}$ and $A_2 = 1/G_*$, the action containing power law terms in R is obtained:

$$\frac{R - 2\Lambda_k}{16\pi G_k} = \frac{R}{16\pi G_*} - \frac{1}{\alpha+1} \frac{C}{R^{\alpha}} - \rho_{\Lambda}^{*} \,. \tag{20}$$

It is interesting to observe that for $A_1 = 0$ and $A_2 = 0$, the effective action of the form $\sim R^{-\alpha}$ is obtained.

5 Scale setting at fixed points - universality in effective modified gravity theories

In the asymptotic safety program [23] (see also [24] for a recent review), an alternative to renormalizability of QFT theories, the existence of non-gaussian fixed point plays a key role. At non-gaussian fixed points (NGFP) the scaling of parameters in the actions, such as G_k and Λ_k, with the cutoff k is determined entirely from dimensionality of these parameters. Next we present our results that the scale-setting procedure results in modified gravity theories with some universal properties.

5.1 Non-gaussian fixed point in Einstein-Hilbert truncation

In Einstein-Hilbert truncation (2) the scaling of the Newton coupling and the cosmological constant at the non-gaussian fixed point are

$$G_k = \frac{g^*}{k^2}\,, \quad \Lambda_k = \lambda^* k^2\,. \tag{21}$$

The scale-setting condition (3) then yields

$$k^2 = \frac{R}{4\lambda^*}\,. \tag{22}$$

This leads to $\Lambda_k = \frac{R}{4}$ and $G_k = \frac{4g^*\lambda^*}{R}$. Inserting these results into (2) yields a modified gravity action

$$S_{\rm EH} = \int d^4x\sqrt{-g}\frac{R^2}{128\pi g^*\lambda^*}\,. \tag{23}$$

Universality features of this action become clear in the absence of matter or in the regime where the influence of matter is negligible. In that case the exact values of constants g^* and λ^* are not important since they just enter a constant $(\frac{1}{128\pi g^*\lambda^*})$ multiplying the entire action and therefore do not affect the equations of motion. The behavior of the system near the non-gaussian fixed point for any g^* and $\lambda^*$is then described by a $f(R)$ modified gravity theory with $f(R) = R^2$. The importance of R^2 effective action, which can be analytically solved [25, 26, 27], was recently stressed in [14, 15], based on Ansatz $k^2 \sim R$. In this paper the relation $k^2 \sim R$ is derived from the scale-setting procedure, together with the coefficient of proportionality.

5.2 Non-gaussian fixed point in more general truncations

For a more general action of the form

$$S = \int d^4x\sqrt{-g}\sum_{m=0}^{n} c_{k,m}R^m\,, \tag{24}$$

the scaling of the coefficients $c_{k,m}$ at a non-gaussian fixed point is $c_{k,m} = a_m k^{4-2m}$. The scale-setting condition (3) then yields

$$\sum_{m=0}^{n}(4-2m)a_m\left(\frac{R}{k^2}\right)^m = 0\,. \tag{25}$$

If we define the polynomial $P(x) = \sum_{m=0}^{n}(4-2m)a_m x^m$ and denote its zeros by x_l, $P(x_l) = 0\,,(l = 1, 2, ..., n)$, the result of the scale-setting procedure is

$$k^2 = \frac{R}{x_l}\,. \tag{26}$$

Inserting the result for k into the action (24) gives

$$S = \int d^4x\sqrt{-g}R^2 \sum_{m=0}^{n} a_m x_l^{m-2} \,. \tag{27}$$

The resulting action is the R^2 action. In the regime where the contribution of matter is negligible, the action exhibits universal properties at the non-gaussian fixed point. Namely, the particular values of the coefficients a_m are immaterial since they all combine into a constant multiplying the entire action ($\sum_{m=0}^{n} a_m x_l^{m-2}$). Another interesting property is that this result is valid for any n, including the case when it becomes arbitrarily large i.e. when the polynomial in the action effectively becomes a series expansion in R.

6 Discussion and conclusions

Gravitational theories with renormalization group corrections all share the problem of determination of the RG scale. Results from the recent literature [14, 15, 21] demonstrate the common notion that the RG scale should be related to some curvature invariant and the Ricci scalar seems to be the first choice. In these approaches, however, the scale is set ad hoc, frequently motivated by qualitative physical arguments. The scale-setting procedure at the level of action introduced in [16] and further elaborated and extended in this paper resolves the scale-setting problem by simple variation of the action over the RG scale. The dynamics of the theory then depends only on the running laws of the parameters of the action and no additional assumptions on the RG scale are needed.

Although the scale-setting procedure reproduces some ad hoc choices for the RG scale proposed in the literature, an important distinction of our results compared to these approaches is that in the scale setting procedure used in this paper the refinement of the running laws (e.g. by the addition of the higher-order terms in the β functions) in general results in different values for the RG scale. In particular, although the scale-setting procedure at the NGFP yields $k^2 \sim R$ as proposed in [14, 15], close to the NGFP the value of the scale deviates from this relationship as higher-order corrections are added.

In Einstein-Hilbert truncation, but also in truncations which are polynomials in R of arbitrary powers, at NGFP the effective action obtained by the scale-setting procedure is described by the action quadratic in R. Furthermore, if at NGFP it were possible to neglect the contributions of matter, the precise values of parameters would become irrelevant. Namely, they all combine into a constant multiplying the entire action and therefore do not affect the dynamics of the system. In that case the universality of the functional form of the effective action is elevated to the universality of the dynamics.

The research presented in this paper is concentrated on the gravitational sector of QFT in the curved space-time. There is no obstacle to extend the scale-setting procedure to matter sector since the principle behind the scale-setting, i.e. the variation of the action over the scale k can be easily applied to couplings and other parameters of the matter Lagrangian. Recent research [28] even suggests that such extension might be an important future step.

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References

[1] I. L. Shapiro and J. Sola, Phys. Lett. B **475** (2000) 236.

[2] I. L. Shapiro and J. Sola, JHEP **0202** (2002) 006.

[3] A. Babic, B. Guberina, R. Horvat and H. Stefancic, Phys. Rev. D **65** (2002) 085002.

[4] A. Babic, B. Guberina, R. Horvat and H. Stefancic, Phys. Rev. D **71** (2005) 124041.

[5] I. L. Shapiro, J. Sola, H. Stefancic, JCAP **0501** (2005) 012.

[6] S. Domazet and H. Stefancic, Phys. Lett. B **703** (2011) 1.

[7] C. Wetterich, Phys. Lett. B **301** (1993) 90.

[8] M. Reuter, Phys. Rev. D **57** (1998) 971.

[9] D. Dou, R. Percacci, Class. Quant. Grav. **15** (1998) 3449.

[10] M. Hindmarsh, D. Litim, C. Rahmede, JCAP **1107** (2011) 019.

[11] A. Bonanno, A. Contillo, Roberto Percacci, Class. Quant. Grav. **28** (2011) 145026.

[12] A. Contillo, Phys. Rev. D **83** (2011) 085016.

[13] A. Contillo, M. Hindmarsh, C. Rahmede, Phys. Rev. D **85** (2012) 043501.

[14] A. Bonanno, arXiv:1203.1962 [hep-th].

[15] M. Hindmarsh and I. D. Saltas, arXiv:1203.3957 [gr-qc].

[16] B. Koch and I. Ramirez, Class. Quant. Grav. **28** (2011) 055008.

[17] C. Ahn, C. Kim, E. V. Linder, Phys. Lett. B **704** (2011) 10.

[18] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rept. **513** (2012) 1.

[19] I. L. Shapiro, Class. Quant. Grav. **25** (2008) 103001.

[20] T. Biswas, E. Gerwick, T. Koivisto, A. Mazumdar, Phys. Rev. Lett. **108** (2012) 031101.

[21] A. V. Frolov and J.-Q. Guo, arXiv:1101.4995 [astro-ph.CO].

[22] S. M. Carroll, V. Duvvuri, M. Trodden, M. S. Turner, Phys. Rev. D **70** (2004) 043528; S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden, M. S. Turner, Phys. Rev. D **71** (2005) 063513.

[23] S. Weinberg in *General Relativity: An Einstein centenary survey*, edited by S. W. Hawking and W. Israel, Cambridge University Press (1979), 790-831.

[24] R. Percacci, arXiv:1110.6389 [hep-th].

[25] T. Clifton, Class. Quant. Grav. **24** (2007) 5073.

[26] S. Domazet, V. Radovanovic, M. Simonovic and H. Stefancic, arXiv:1203.5220 [gr-qc].

[27] J. D. Barrow, J. Middleton, Phys. Rev. D **75** (2007) 123515.

[28] A. Eichhorn, arXiv:1204.0965 [gr-qc].